

Cosmological perturbations in non-local higher-derivative gravity

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Abstract. We study cosmological perturbations in a non-local higher-derivative model of gravity introduced by Biswas, Mazumdar and Siegel. We extend previous work, which had focused on classical scalar perturbations around a cosine hyperbolic bounce solution, in three ways. First, we point out the existence of a Starobinsky solution in this model, which is more attractive from a phenomenological point of view (even though it has no bounce). Second, we study classical vector and tensor perturbations. Third, we show how to quantize scalar and tensor perturbations in a de Sitter phase (for choices of parameters such that the model is ghost-free). Our results show that the model is well-behaved at this level, and are very similar to corresponding results in local $f(R)$ models. In particular, for the Starobinsky solution of non-local higher-derivative gravity, we find the same tensor-to-scalar ratio as for the conventional Starobinsky model.

Keywords: modified gravity, cosmological perturbation theory, quantum cosmology, inflation

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1 Introduction

The resolution of cosmological singularities is an outstanding question in theoretical cosmology. Inflationary models are often geodesically incomplete [1], so one needs to understand what happens at (or replaces) the singularity in order to explain how the universe emerged in a state that enabled inflation. In alternatives to inflation that involve a transition from a contracting universe into an expanding one, understanding this (possibly singular) transition is even more crucial, since it determines how cosmological perturbations generated in the contracting phase match onto perturbations in the expanding phase.

The expectation that an ultraviolet (UV) complete theory of quantum gravity should resolve cosmological singularities has motivated the construction of many string theory models; see for instance [2–15] for reviews. Even though interesting mechanisms for singularity resolution have emerged, they are often plagued by technical problems or ambiguities. It remains therefore an open question whether and how string theory resolves cosmological singularities.

An alternative approach is to explore bottom-up models inspired by string theory, and a lot of work has happened along these lines. In particular, it is known that the UV properties of general relativity can be ameliorated by higher derivative corrections. A well-studied class of models is $f(R)$ gravity [16–20], which has the advantage that the equations governing

cosmological perturbations are tractable. However, models with terms up to a certain order in derivatives generically contain ghosts (negative-norm or negative-energy states). This problem has been circumvented in a class of higher derivative *non-local* models of gravity introduced in [21]. These models, which we will review in section 2, are inspired by infinite-derivative structures appearing in string field theory, and should be viewed as effective actions (which indeed tend to be non-local). In [21], a ghost-free model was constructed and an exact bouncing, spatially flat FLRW solution was found.¹ This solution represents a universe bouncing between contracting and expanding de Sitter phases supported by a cosmological constant. Interpreting this cosmological constant as a simplified model for the energy density driving inflation, the bouncing solution can model the inflationary phase of our universe. Of course it should really be supplemented by a graceful exit into conventional big bang cosmology, but so far this can only be achieved in an approximate way [21].

Among the most important predictions of inflationary cosmology are nearly scale-invariant spectra of adiabatic density perturbations and gravity waves, so it is very important to investigate how perturbations behave in the model of [21]. A first analysis of scalar perturbations was performed in [23], where the crucial assumption was made that the modification of gravity only affects the background evolution, while the perturbations are governed by Einstein gravity. Perturbations that only depend on time (and therefore model super-Hubble perturbations) were subsequently studied in [24], where it was concluded that in the late-time de Sitter phase all modes decay, except for one constant mode (which was suggested to be relevant for seeding density perturbations). However, the constant mode identified in [24] is really a gauge artifact (it can be removed by a constant rescaling of the spatial coordinates). A careful study of all classical scalar perturbations, with dependence on both time and space (including sub-Hubble modes as well as super-Hubble modes) was undertaken in [25]. This involved the development of non-trivial techniques to deal with the relevant non-local actions. The conclusion of this work is that (at least in appropriate parameter ranges) all modes are non-singular during the bounce, and decay in the late-time de Sitter phase. Since in this simplified model the de Sitter phase is supported by a cosmological constant rather than a dynamical inflaton field, this is the desired result. At late times, the higher-derivative corrections to general relativity are unimportant, so one can expect that including a dynamical inflaton will give rise to a nearly scale-invariant spectrum of adiabatic density perturbations, as it does in general relativity.

A more realistic inflationary solution was found in $R + R^2$ gravity, which is a *local* higher derivative model. It is the Starobinsky solution [26], which has an inflationary phase followed by an exit from inflation, producing a nearly scale invariant spectrum of perturbations. We show that this spacetime also emerges as a solution of non-local gravity. An important difference with the already mentioned bouncing solutions is that in the case of the Starobinsky solution inflation is driven by higher derivative terms while in the known bouncing solutions inflation is driven by a cosmological constant.

One question that was not addressed in [25] is how vector and tensor perturbations behave in these non-local models. We present a simple argument showing that linearized vector and tensor perturbations are governed by *local* evolution equations, which are essentially identical to those of $f(R)$ gravity models and thus much simpler than those of scalar

¹While originally it was believed that the model constructed in [21] is asymptotically-free, the more complete analysis in [22] showed that it is not, but that it is possible to construct an asymptotically free model of gravity by adding additional non-local terms to the action. In the present paper, we will focus on the original model of [21], leaving the possible generalization of our results to asymptotically free models to future work.

perturbations. We compute vector and tensor perturbations in the bouncing solution of [21], thereby completing the analysis of classical cosmological perturbations in this model. We also comment on the growth of anisotropies in a contracting phase. In addition, we compute the behavior of vector and tensor modes around the Starobinsky solution.

Another question we address in the present paper is how to quantize cosmological perturbations. Local $f(R)$ gravity models give rise to only one physical mode in the scalar sector of perturbations. Generically, non-local models with higher derivatives give rise to many degrees of freedom. However, by imposing certain restrictions on the non-local operator involved, one can ensure that the model has only one physical excitation (as we have in local $f(R)$ Lagrangians). We quantize scalar and tensor perturbations in the de Sitter regime of the Starobinsky solution and compute their power spectra.

The remainder of this paper is organized as follows. In section 2, we review the model of [21], considering the bouncing solution and presenting the Starobinsky solution in this framework. In section 3 we revisit scalar perturbations and derive the general equations for vector and tensor perturbations. We apply them to the bouncing and Starobinsky backgrounds. In section 4 we discuss quantum perturbations. We conclude with a summary and outlook in section 5.

2 The model and background solutions

2.1 The model

The string-inspired higher derivative non-local model of gravity we will consider is described by the action [21]

$$S = \int d^4x \sqrt{-g} \left(\frac{M_P^2}{2} R + \frac{\lambda}{2} R \mathcal{F}(\square) R - \Lambda + \mathcal{L}_M \right). \quad (2.1)$$

Here M_P is the Planck mass, Λ is a cosmological constant, \mathcal{L}_M is a matter Lagrangian² and λ is a dimensionless parameter measuring the effect of the $O(R^2)$ corrections. We work in four dimensions and also limit ourselves to $O(R^2)$ corrections.

The operator $\mathcal{F}(\square)$, which is inspired by structures appearing in string field theory, is the ingredient making the difference with other modifications of gravity. It is an analytic function with real coefficients of the d'Alembertian $\square = \nabla^\mu \nabla_\mu$:

$$\mathcal{F}(\square) = \sum_{n \geq 0} f_n \square^n / M_*^{2n}. \quad (2.2)$$

The new mass scale M_* determines the characteristic scale of the modification of gravity. Hereafter we absorb M_* in the coefficients f_n .

The equations of motion derived from the action (2.1) are

$$\begin{aligned} [M_P^2 + 2\lambda \mathcal{F}(\square) R] G_\nu^\mu &= -\frac{\lambda}{2} R \mathcal{F}(\square) R \delta_\nu^\mu + 2\lambda (\nabla^\mu \partial_\nu - \delta_\nu^\mu \square) \mathcal{F}(\square) R \\ &\quad + \lambda \mathcal{K}_\nu^\mu - \frac{\lambda}{2} \delta_\nu^\mu (\mathcal{K}_\sigma^\sigma + \bar{\mathcal{K}}) + T_\nu^\mu - \Lambda \delta_\nu^\mu, \end{aligned} \quad (2.3)$$

²In this article we will focus on cases in which \mathcal{L}_M contains at most radiation.

where we have defined

$$\mathcal{K}_\nu^\mu = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \partial^\mu R^{(l)} \partial_\nu R^{(n-l-1)}, \quad \bar{\mathcal{K}} = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} R^{(l)} R^{(n-l)}. \quad (2.4)$$

Here G_ν^μ is the Einstein tensor, $R^{(n)} = \square^n R$ and T_ν^μ is the matter energy momentum tensor. The trace equation reads

$$-M_P^2 R = T - 4\Lambda - 6\lambda \square \mathcal{F}(\square) R - \lambda(\mathcal{K} + 2\bar{\mathcal{K}}), \quad (2.5)$$

where quantities without indices denote the trace. This equation is clearly much simpler than (2.3).

2.2 Ansatz

Particular background solutions were found by employing the ansatz

$$\square R = r_1 R + r_2, \quad (2.6)$$

where $r_{1,2}$ are constants. One can then easily compute the $\square^n R$ expression recurrently. Clearly, this simplifies the equations of motion considerably but it is apparently non-trivial to satisfy the ansatz itself. Substitution of the ansatz in the trace of the Einstein equations (2.5) yields three types of terms which are constant, linear and quadratic in curvature, respectively. The following are three relations to cancel each structure separately:

$$\mathcal{F}^{(1)}(r_1) = 0, \quad \frac{r_2}{r_1} = -\frac{M_P^2 - 6\lambda \mathcal{F}_1 r_1}{2\lambda[\mathcal{F}_1 - f_0]}, \quad \Lambda = -\frac{r_2 M_P^2}{4r_1}, \quad (2.7)$$

where $\mathcal{F}_1 = \mathcal{F}(r_1)$ and $\mathcal{F}^{(1)}$ is the first derivative with respect to its argument. Using these expressions, the equations of motion after substituting the ansatz are local; we refer to them as Einstein-Schmidt equations:

$$2\lambda \mathcal{F}_1 (R + 3r_1) G_\nu^\mu = T_\nu^\mu + 2\lambda \mathcal{F}_1 \left[g^{\mu\rho} \nabla_\rho \partial_\nu R - \frac{1}{4} \delta_\nu^\mu (R^2 + 4r_1 R + r_2) \right]. \quad (2.8)$$

The same local equations of motion can also be obtained from an $R + R^2$ gravity model with action

$$S = \int d^4x \sqrt{-g} \left(R + \frac{1}{6r_1} R^2 - 2\Lambda_{f(R)} + 2\mathcal{L}_{M,f(R)} \right), \quad (2.9)$$

where

$$\Lambda_{f(R)} = \frac{-r_2}{2r_1} = \frac{2\Lambda}{M_P^2} \quad \text{and} \quad \mathcal{L}_{M,f(R)} = \frac{\mathcal{L}_M}{6\lambda \mathcal{F}_1 r_1}. \quad (2.10)$$

We focus here on background solutions of flat FLRW type, so we look for metrics of the form

$$g_{\mu\nu} = \text{diag}(-1, a^2(t), a^2(t), a^2(t)), \quad (2.11)$$

where $a(t)$ is the scale factor. Solutions of the local equations of motion that also satisfy the ansatz are solutions of the non-local model. Therefore any solution of $R + R^2$ gravity that solves the ansatz is a solution of non-local higher derivative gravity (2.1). Note that after using the ansatz the model has no smooth GR limit $\lambda \rightarrow 0$ anymore.

2.3 Bouncing solution

One can show [21, 25] that

$$a(t) = a_0 \cosh(\omega t) \quad (2.12)$$

is a solution to the ansatz relation with $r_1 = 2\omega^2$ and $r_2 = -6r_1^2$. From e.g. [21] we know that a cosine hyperbolic bounce is a solution of $R + R^2$ gravity if some ghostlike (i.e. $\rho_{f(R)} < 0$) radiation is added. It is therefore also a solution of the non-local model with radiation. From analyzing the 00 component of (2.8) the energy density of the traceless radiation in the non-local model is found to be

$$\rho_0 = -\frac{27}{2}\lambda\mathcal{F}_1 r_1^2. \quad (2.13)$$

This implies $\mathcal{F}_1 < 0$ in order to avoid ghost components. Comparison of ρ_0 with $\rho_{0f(R)}$ by making use of (2.10) shows that positive radiation in the non-local model indeed corresponds to ghostlike radiation in the local $R + R^2$ model.

The cosine hyperbolic bounce is a non-singular solution which reaches a de Sitter phase asymptotically at $t \rightarrow \pm\infty$. The bouncing solution is therefore said to reach a late time inflating phase. This inflationary phase however has no graceful exit and therefore only serves as a toy model.

2.4 Starobinsky solution

Now we present an inflationary solution of the non-local model which at some point in time stops inflating. The scale factor

$$a(t) = a_0 \sqrt{t_s - t} e^{\sigma(t_s - t)^2} \quad (2.14)$$

is an approximate solution of $R + R^2$ gravity without cosmological constant at early times when $|t| \ll t_s$. It is known as the Starobinsky solution. When $|t| \ll t_s$ and moreover also

$$|\sigma t_s^2| \gg \frac{1}{4}, \quad (2.15)$$

the Starobinsky solution can be approximated by a de Sitter phase. The condition (2.15) guarantees a nearly constant Hubble rate. As is clear from (2.9), this corresponds to the regime where the R^2 correction to gravity dominates. The parameters σ and t_s can be determined from observations, e.g., from the normalization of the scalar power spectrum and from its spectral tilt.

If a cosmological constant is added, it can be checked that (2.14) becomes an exact solution to the equations of motion of $R + R^2$ gravity for all times. It can also be checked that this background solves the ansatz (2.6) too. So we conclude that it is also a solution of the non-local model. The coefficients for which the ansatz is satisfied, are $r_1 = -12\sigma$ and $r_2 = 192\sigma^2 = \frac{4}{3}r_1^2$. The condition (2.15) can then be expressed as $12H^2/r_1 \gg 1$. Notice that exponential expansion is only reached for $\sigma < 0$. This means that $r_1 > 0$ and $\Lambda < 0$. In this case, the universe with scale factor (2.14) undergoes inflation when $t \ll t_s$, after which it has a graceful exit. If one were to consider the scale factor at later times, one would find that the universe reaches a maximum size at $t_s - t = \sqrt{-1/(4\sigma)}$, after which it contracts due to the negative cosmological constant until it eventually hits a singularity at $t = t_s$.

To see whether radiation is needed for having the Starobinsky solution, the 00 component of the Einstein-Schmidt equations (2.8) is checked. It turns out that the 00 equation is satisfied without any radiation. When the Starobinsky model is coupled to particle physics

models, radiation should be produced during reheating. This goes beyond the scope of the present article, in which we will focus on observables that can be computed in the inflationary phase.

3 Classical perturbations

In this section we analyze the behavior of classical perturbations in non-local higher derivative gravity. In particular, we discuss their behavior around the cosine hyperbolic bounce and in the Starobinsky background. Scalar perturbations in the bouncing solution were already discussed in [25]. We briefly review their analysis and extend it to vector and tensor perturbations, as well as to the Starobinsky solution.

3.1 Scalar perturbations

Scalar perturbations are introduced as

$$ds^2 = a(\tau)^2 \left[-(1 + 2\phi)d\tau^2 - 2\partial_i\beta d\tau dx^i + ((1 - 2\psi)\delta_{ij} + 2\partial_i\partial_j\gamma)dx^i dx^j \right], \quad (3.1)$$

where τ is the conformal time related to the cosmic one as $a(\tau)d\tau = dt$.

It is practical to use gauge-invariant variables to avoid gauge fixing issues. These variables are two Bardeen potentials, which fully determine the scalar perturbations of the metric tensor [16, 27, 28]

$$\Phi = \phi - \frac{1}{a}(a\vartheta)' = \phi - \dot{\chi}, \quad \Psi = \psi + \mathcal{H}\vartheta = \psi + H\chi, \quad (3.2)$$

where $\chi = a\beta + a^2\dot{\gamma}$, $\vartheta = \beta + \gamma'$, $\mathcal{H}(\tau) = a'/a$, and differentiation with respect to the conformal time τ is denoted by a prime.

Before considering these perturbations in non-local higher derivative gravity [25], we first briefly summarize what inflationary theories typically tell us about scalar metric perturbations.

Standard inflation leads to an exponential expansion of the physical wavelengths of fluctuations and thus effectively brings perturbations that are initially inside the horizon, to the outside where they get frozen in. Inside the horizon, sub-Hubble fluctuations are present due to quantum fluctuations of the inflaton field. These sub-Hubble metric fluctuations described by the Bardeen potential are oscillatory with a constant amplitude. During inflation they are carried outside the horizon and once this has happened perturbations are frozen in and the Bardeen potential is approximately a constant.

$f(R)$ gravity models provide another way of generating inflation. Because of a conformal equivalence between $f(R)$ gravity and normal Einstein gravity with addition of scalar field matter, metric perturbations will behave more or less in the same way as in standard inflation. This means that scalar perturbations are also oscillating when they are sub-Hubble but after crossing the horizon they get frozen in and become approximately constant on super-Hubble scales.

Now we turn our attention to the behavior in the non-local model. Technically, scalar perturbations are subject to very complicated equations and the only tractable configurations are those satisfying the ansatz (2.6) at the background level. We focus here on two particular solutions, namely the bouncing and the Starobinsky solution. Both have a de Sitter phase, which is the relevant region we will focus on. Stability under scalar perturbations near the bounce was already addressed in [25] and also the behavior during a de

Sitter phase was presented. Here we only present the key results of [25] and focus on the physical interpretation.

We immediately write down the two coupled closed equations which read

$$\mathcal{P}\zeta = 0, \quad (3.3)$$

$$\frac{\mathcal{F}(\square) - \mathcal{F}_1}{\square_B - r_1} \zeta + \mathcal{F}_1 [\delta R_{GI} + (R_B + 3r_1)(\Phi - \Psi)] = 0, \quad (3.4)$$

where

$$\zeta = \delta \square R_B + (\square_B - r_1) \delta R_{GI}, \quad (3.5)$$

$$\begin{aligned} \mathcal{P} \equiv & [\partial^\mu R_B \partial_\mu + 2(r_1 R_B + r_2)] \frac{\mathcal{F}(\square_B) - \mathcal{F}_1}{(\square_B - r_1)^2} + 3\mathcal{F}(\square_B) \\ & + (R_B + 3r_1) \frac{\mathcal{F}(\square_B) - \mathcal{F}_1}{\square_B - r_1}, \end{aligned} \quad (3.6)$$

$$\square_B = -\frac{1}{a^2} \partial_\tau^2 - 2\frac{a'}{a^3} \partial_\tau - \frac{k^2}{a^2} = -\partial_t^2 - 3\frac{\dot{a}}{a} \partial_t - \frac{k^2}{a^2} \quad (3.7)$$

$$\delta \square = \frac{1}{a^2} \left[2\Phi \left(\partial_\tau^2 + 2\frac{a'}{a} \partial_\tau \right) + (\Phi' + 3\Psi') \partial_\tau \right], \quad (3.8)$$

$$\delta R_{GI} = 6\square_B \Psi - 2R_B \Phi - 6\frac{a'}{a^3} (\Phi' + \Psi') + 2\frac{k^2}{a^2} (\Phi + \Psi). \quad (3.9)$$

Here the first equation is the perturbation of the trace equation and the second comes from considering the ij component of the Einstein equations when $i \neq j$. Essentially, ζ is the variation of the ansatz (2.6). This ansatz is a crucial element in finding background solutions of the non-local model. However, being an ansatz it only holds at the background level and will in general be perturbed. Here δR_{GI} is a gauge invariant quantity defined as $\delta R_{GI} = \delta R - R'_B(\beta + \gamma')$. If $\zeta = 0$, then the ansatz is unperturbed and the equations of motion are local as in $R + R^2$ gravity even at the perturbed level. Therefore we say that ζ denotes whether perturbations are influenced by non-localities or not. k is the length of the comoving spatial momentum.

For the full picture two more equations can be derived which relate metric perturbations to energy density and velocity perturbations of the fluid. They are the $0i$ and 00 component of the Einstein equations and were computed in [25]. Both in GR and in $R + R^2$ gravity the perturbed trace equation would be identically zero. Moreover in GR the ij component reduces to $\Phi = \Psi$. The subscript B designates background quantities.

We are mainly interested here in the behavior during a de Sitter phase because this mimics an inflationary situation. As derived in [25] the variation of the trace under scalar perturbations can then be written as

$$(\square_B - r_1) \mathcal{W}(\square_B) \delta R_{GI} = 0, \quad (3.10)$$

where $\mathcal{W}(\square)$ is defined as

$$\mathcal{W}(\square) \equiv 3\mathcal{F}(\square) + (R_B + 3r_1) \frac{\mathcal{F}(\square) - \mathcal{F}_1}{\square - r_1}. \quad (3.11)$$

The solution δR_{GI} can be written as a linear superposition of eigenmodes of the d'Alembertian operator with corresponding eigenvalues ω_i^2 that are roots of

$\mathcal{W}(\omega^2)(\omega^2 - r_1) = 0$. The eigenmodes $\delta R_{GI}^{(i)}$ satisfy

$$(\square_B - \omega_i^2) \delta R_{GI}^{(i)} = 0. \quad (3.12)$$

The most general solution to this equation can be written in terms of the Bessel function J_ν and the Neumann function Y_ν as

$$\delta R_{GI}^{(i)} = (-k\tau)^{\frac{3}{2}} [d_{1k} J_\nu(-k\tau) + d_{2k} Y_\nu(-k\tau)] \quad (3.13)$$

with d_{1k} and d_{2k} constant in time and

$$\nu = \sqrt{\frac{9}{4} - \frac{\omega_i^2}{H^2}}. \quad (3.14)$$

3.1.1 Bouncing solution

In case of the cosine hyperbolic bounce, the de Sitter regime is reached at late times, consider the long wavelength modes. At late times when the cosmological constant dominates and causes the exponential expansion of the universe. The non-local modification to the action (2.1) has no influence and the behavior of perturbations should be the same as in Einstein gravity with a cosmological constant. In that case perturbations are decaying. Growing modes in this regime would make the late-time expanding phase unstable [25]. So we demand that during this late-time de Sitter phase scalar modes are decaying which is only the case if every root of $\mathcal{W}(\square)$ is such that the real part ν_R of ν satisfies

$$|\nu_R| < \frac{3}{2}. \quad (3.15)$$

It can be checked that also the mode $\omega_i^2 = r_1$ which is always present and which corresponds to the mode of $R + R^2$ gravity,³ is decaying.

3.1.2 Starobinsky solution

The Starobinsky solution reaches an exponentially expanding phase at early times caused by the non-local higher derivative term in the action (2.1). By analyzing (3.12) in the short wavelength limit it is found that, as in $R + R^2$ gravity, short-wavelength perturbations are oscillatory and decay as $1/a$. Once the perturbations reach the long wavelength regime they should freeze out and become approximately constant. The roots of $\mathcal{W}(\square)$ should therefore satisfy

$$|\nu_R| \approx \frac{3}{2}. \quad (3.16)$$

Extra decaying modes will not influence the phenomenology so the modes for which $|\nu_R| < 3/2$ are also allowed but they are not interesting.

The mode corresponding to $\omega_i^2 = r_1$ also exhibits $\nu_R \approx 3/2$. This follows from considering the Starobinsky solution in the regime when it is approximately de Sitter, taking $r_1 = -12\sigma$ and applying the corresponding limits as discussed in section 2.4. In this regime, we have

$$a(t) \approx a_0 \sqrt{t_s} e^{-2\sigma t_s t} \text{ and } H \approx -2\sigma t_s. \quad (3.17)$$

³This is the mode for which the ansatz is unperturbed.

This yields for (3.14)

$$\nu \approx \sqrt{\frac{9}{4} + \frac{3}{\sigma t_s^2}} \rightarrow \frac{3}{2}, \quad (3.18)$$

provided that (2.15) holds. Taking into account the small-argument asymptotic behaviors $J_{3/2}(x) \sim x^{3/2}$ and $Y_{3/2}(x) \sim x^{-3/2}$, we find that the mode $\omega_i^2 = r_1$ is approximately a constant as expected.

The two remaining Einstein equations, namely the 00 and 0*i* components, in general provide conditions involving the energy density and velocity perturbation of matter. Since the Starobinsky solution is obtained in a model without matter or radiation, these equations simply impose additional conditions on the Bardeen potentials. Denoting $u = \mathcal{F}(\square_B) \delta R_{GI}$, the remaining Einstein equations in a de Sitter limit are

$$\begin{aligned} -6\lambda \frac{\mathcal{H}^2}{a^2} u + 4\lambda \mathcal{F}_1 (R_B + 3r_1) \left(\frac{k^2}{a^2} \Psi + 3 \frac{\mathcal{H}}{a^2} (\Psi' + \mathcal{H}\Phi) \right) \\ + 2\lambda \square_B u + \lambda R_B u + \frac{2\lambda}{a^2} (u'' - \mathcal{H}u') = 0, \end{aligned} \quad (3.19)$$

$$2\mathcal{F}_1 (R_B + 3r_1) (\Psi' + \mathcal{H}\Phi) - u' + \mathcal{H}u = 0. \quad (3.20)$$

Notice that using this notation, the last equation contains at most first order derivatives and can therefore serve as a constraint on the initial data for the Bardeen potentials.

3.2 Vector perturbations

Vector perturbations in GR die out during inflation and are therefore unimportant. As long as there is no additional rotational anisotropy in the matter tensor, angular momentum is conserved, which leads to a decaying solution of the metric perturbation. The same holds for $f(R)$ gravity, but need not be true in more general modified gravity.

Vector modes are introduced as

$$ds^2 = -a^2(\tau) [d\tau^2 - 2b_i dx^i d\tau + (\delta_{ij} + \partial_i c_j + \partial_j c_i) dx^i dx^j]. \quad (3.21)$$

By definition the vector modes b_i, c_i are transverse, i.e., $\partial_i b^i = 0$ and $\partial_i c^i = 0$. These vector modes are related through a gauge transformation and only two independent gauge invariant degrees of freedom remain, namely $\Psi_i = b_i + c'_i$ [16, 17, 27].

At the linearized level, $\delta R_{GI} = 0$ and $\delta \square f(t) = 0$ under vector perturbations. That is because R as well as $f(t)$ are scalars while vector and scalar perturbations do not mix at linear order. This leads to

$$[M_P^2 + 2\lambda \mathcal{F}(\square_B) R_B] \left(-\frac{1}{2a^2} \Delta \Psi_i \right) = \delta T_i^0, \quad (3.22)$$

where $\delta T_i^0 = (\rho_B + p_B) \delta u_i / a$, with δu_i being a perturbation of the fluid 4-velocity. From $\nabla_\mu T_i^\mu = 0$, it follows that $[a^3 (\rho_B + p_B) \delta u_i]' = 0$. This expresses angular momentum conservation, and implies that $\delta T_i^0 = L_k / a^4$, with L_k is the angular momentum of the rotational mode. Then in Fourier space vector modes can be written as

$$k^2 \Psi_i = \frac{2L_k}{a^2 (M_P^2 + 2\lambda \mathcal{F}(\square_B) R_B)}, \quad (3.23)$$

which reduces to the GR result in the limit of $\lambda \rightarrow 0$.

Analyzing the behavior of the latter expression on a class of solutions satisfying the ansatz (2.6) we find that

$$k^2 \Psi_i = \frac{L_k}{\lambda \mathcal{F}_1 a^2 (R_B + 3r_1)}. \quad (3.24)$$

A vector mode thus behaves in exactly the same way as in $R + R^2$ gravity. This could have been anticipated, because the conditions $\delta R_{GI} = \delta \square R_B = 0$ imply that the ansatz is unperturbed. The classically equivalent $R + R^2$ model of gravity thus remains valid at the perturbed level and vector modes behave as in local higher derivative gravity [17].

3.2.1 Bouncing solution

Now we specialize to the bounce solution (2.12), and study the behavior of Ψ_i in (3.24), both near the bounce and in the late-time de Sitter phase. We are interested in answering the following two questions: do we recover the GR limit at late times? And do vector perturbation pass the bounce smoothly? Below we will show that vector perturbations at late times do behave as in GR. Secondly, in order for the bouncing mechanism not to be destroyed, perturbations should be finite near the bounce. Since the bounce itself is non-singular, this will indeed be the case.

First we focus on the late-time behavior. When $t \gg \omega^{-1}$ one enters the de Sitter phase and the modification term in (3.24) becomes negligible (see also [21]). In this regime the scale factor is given by $a(t) \approx a_0 e^{\omega t}/2$. From $H(t) = \omega \tanh(\omega t)$ it follows that at late times H is approximately a constant, which in turn implies that R_B is a constant. So when $t \gg \omega^{-1}$ the behavior of vector perturbations is $\Psi_i \propto 1/a^2$, which is exactly the result one can find in GR (see e.g. [17, 29]).

Near the bounce then, $a \rightarrow a_0$ and $H \rightarrow 0$. Expressed in cosmic time $R_B = 6(\dot{H} + 2H^2)$. Since H is very small near the bounce, we neglect the H^2 term. Since H is smooth near the bounce, its time derivative is well defined and finite, and R_B is certainly finite. In fact one can easily show that for the cosine hyperbolic bounce $R_B \approx 6\omega^2$. Since R_B is decreasing in the contracting phase while it is increasing in the expanding phase, we can conclude that a vector mode starting out in the contracting phase, grows near the bounce, reaches a maximal size at the bounce and then decays again.

3.2.2 Starobinsky solution

Around an inflationary solution the analysis is equivalent to the late-time limit of the bouncing solution. Approximating the Starobinsky solution as a de Sitter one, the constancy of background curvature implies that the vector modes (3.24) decay like $1/a^2$ in non-local higher derivative gravity. In an inflationary context non-local gravity therefore predicts no cosmological vector modes.

3.3 Tensor perturbations

Unlike scalar perturbations, in standard inflation tensor modes do not couple to the inflaton field. However, the evolution of tensor modes is described by a covariant Klein-Gordon equation, which means that inside the horizon they are oscillatory solutions. On these small scales one can approximate the metric with a local Minkowski metric so that indeed the sub-Hubble tensor modes describe free gravitational waves. During inflation the tensor modes are carried outside the horizon such that on super-Hubble scales they get frozen in.

Tensor perturbations h_{ij} are introduced as

$$ds^2 = -a^2(\tau) d\tau^2 + a^2(\tau) (\delta_{ij} + 2h_{ij}(\mathbf{x}, \tau)) dx^i dx^j. \quad (3.25)$$

These modes are by definition symmetric, traceless and transverse, i.e. $h_i^i = 0$ and $\partial_i h_j^i = 0$, so that one ends up with 2 independent modes [16, 17, 27]. Tensor modes are gauge invariant.

For the same reason discussed above for vector modes, $\delta R = \delta \square f(t) = 0$ under tensor perturbations. The equation of motion becomes

$$h_{ij}'' + \left(2\mathcal{H} + \frac{2\lambda (\mathcal{F}(\square_B) R_B)'}{M_P^2 + 2\lambda \mathcal{F}(\square_B) R_B} \right) h_{ij}' + k^2 h_{ij} = 0, \quad (3.26)$$

which again reduces to the GR equation in the limit $\lambda \rightarrow 0$. Turning to background solutions satisfying the ansatz (2.6) we have in cosmic time

$$\ddot{h}_{ij} + \left(3H + \frac{\dot{R}_B}{R_B + 3r_1} \right) \dot{h}_{ij} + \frac{k^2}{a^2} h_{ij} = 0. \quad (3.27)$$

This is the same evolution as in $R + R^2$ gravity [17].

3.3.1 Bouncing solution

In the late-time regime of the bouncing solution (2.12), the k^2/a^2 term in equation (3.27) can be neglected because the scale factor becomes exponentially large. Since the scalar curvature is constant at late times one can drop the $\dot{R}_B/(R_B + 3r_1)$ term in eq. (3.27) as well. With these approximations eq. (3.27) can easily be integrated

$$h_{ij}(k, t) = c_{ij}(k) - d_{ij}(k) \int \frac{dt}{a^3}, \quad (3.28)$$

which is exactly the result one finds in GR [17, 29].

Near the bounce, we can expand equation (3.27) up to linear order in t resulting in

$$\ddot{h}_{ij} + 4\omega^2 t \dot{h}_{ij} + \frac{k^2}{a_0^2} h_{ij} = 0. \quad (3.29)$$

In order to get a picture of how tensor perturbations behave near the bounce, we decompose $h = \alpha \kappa$ and write $\frac{d}{dt} \ln [a^3 (R_B + 3r_1)] = v$. Then (3.27) becomes

$$\frac{\ddot{\alpha}}{\alpha} + \frac{\dot{\kappa}}{\kappa} + \frac{\dot{\alpha}}{\alpha} \left(v + 2\frac{\dot{\kappa}}{\kappa} \right) + v \frac{\dot{\kappa}}{\kappa} + \frac{k^2}{a^2} = 0. \quad (3.30)$$

We fix κ by the requirement

$$v + 2\frac{\dot{\kappa}}{\kappa} = 0 \Rightarrow \kappa(t) = e^{-\frac{1}{2} \int v dt} = \frac{1}{\sqrt{a^3 (R_B + 3r_1)}} \quad (3.31)$$

The resulting κ is regular around the bounce. Thanks to the choice of κ we are left with the second order differential equation on α only

$$\ddot{\alpha} + \left(\frac{k^2}{a^2} - \frac{\dot{v}}{2} - \frac{v^2}{4} \right) \alpha = 0, \quad (3.32)$$

which is moreover a Schrödinger type equation with zero energy and potential

$$V = \frac{\dot{v}}{2} + \frac{v^2}{4} - \frac{k^2}{a^2}. \quad (3.33)$$

Near the bounce $v = 4\omega^2 t$ so that the potential becomes just $V = (2a_0^2\omega^2 - k^2)/a_0^2$. The evolution equation (3.32) for α becomes a (possibly inverted) harmonic oscillator for which the evolution is described by

$$\alpha(t) = Ae^{\sqrt{V}t} + Be^{-\sqrt{V}t}. \quad (3.34)$$

where A, B are integration constants. For small wavenumbers $k^2 < 2a_0^2\omega^2$, we get a linear combination of an exponentially growing and a decaying mode. They both pass the bounce point at $t = 0$ smoothly. Tensor perturbations with shorter wavelengths for which $k^2 > 2a_0^2\omega^2$ become oscillatory near the bounce. This condition on oscillatory solutions only holds near the bounce and has nothing to do with the definition of sub-Hubble perturbations. Indeed, at zeroth order the Hubble radius H^{-1} near the bounce is infinite such that all perturbations are inside the Hubble radius. We can conclude that a tensor mode generated in the contracting phase will start oscillating near the bounce, pass the bounce smoothly and become constant at late times.

3.3.2 Starobinsky solution

Again the analysis of tensor modes around the Starobinsky background is comparable to the late-time analysis around the bouncing solution. However, the idea now is that perturbations start out sub-Hubble at the beginning of inflation. The variation of curvature is negligible during a quasi de Sitter stage and inside the Hubble radius the friction term in (3.27) can be neglected. In conformal time the evolution equation reduces to that of a harmonic oscillator. Once the perturbations reach the long wavelength regime $k \ll Ha$ the friction term dominates and (3.27) is solved by (3.28). As in the late time regime of the cosine hyperbolic background, tensor modes freeze out. Like standard inflation or $R+R^2$ gravity, non-local higher derivative gravity predicts the generation of cosmological gravitational waves.

3.4 Special case: homogeneous anisotropic perturbations

A special class of homogeneous diagonal perturbations of the metric, is introduced as $ds^2 = -dt^2 + a^2 e^{2\eta_i} dx_i^2$ with $\sum_i \eta_i = 0$. This is a metric of the type Bianchi I which describes an anisotropic universe and is a test-bed for the analysis of anisotropic perturbations. In GR the η_i appear as a new effective matter component in the Einstein equations energy whose energy scales as a^{-6} , leading to problems in a contracting phase (see for instance [30]). In the context of a non-singular bounce, this growth of anisotropies leads to a fine tuning problem on the initial perturbations in the contracting phase. We now study this issue in the context of non-local gravity.

In the perturbative regime ($\eta_i \ll 1$), the metric becomes

$$ds^2 = -dt^2 + a^2(1 + 2\eta_i)dx_i^2, \quad (3.35)$$

and one can try to identify the η_i within the general formalism of linear perturbations introduced before. The question is whether they belong to the scalar, vector or tensor sector. However it should be noted that although one can always split a 4×4 matrix into a spin 0, spin 1 and a spin 2 piece, the explicit distinction between pure scalar modes, vector and tensor modes is ill-defined if $k = 0$ because of the trivially vanishing spatial derivatives. Still,

one can treat the η_i modes as if they were inhomogeneous, go to Fourier space, and then take the limit $k \rightarrow 0$. Since the η_i modes are diagonal and traceless, it follows that there are two independent modes. One of them will contain a longitudinal part while the other one will be completely transverse. If the η_i modes are written in vector notation as $\vec{\eta} = (\eta_1, \eta_2, \eta_3)$, then the tracelessness condition causes the most general mode $\vec{\eta}$ to be a linear combination of a $(1, 1, -2)$ and a $(1, -1, 0)$ mode. These correspond respectively to a scalar and a tensor mode. From the form of the line element we see that all η_i should satisfy the same evolution equation. Both the scalar and the tensor mode should then also satisfy this same equation in the limit $k \rightarrow 0$.

To derive the equation in question we note that comparing the metric (3.35) with the general perturbed line element (the combination of (3.1), (3.21), (3.25)) it follows that the only non-zero modes are γ and h_{ii} . If η_i is a scalar mode, it can be written as $\eta_i = \partial_i^2 \gamma$ and from $\sum_i \eta_i = 0$ it follows that $\Delta \gamma = 0$ or in Fourier space $k^2 = 0$. It is the only non-zero scalar mode so the gauge invariant potentials can be written as $\Phi = -\frac{1}{2} \mathcal{H} \gamma' - \frac{1}{2} \gamma''$ and $\Psi = \frac{1}{2} \mathcal{H} \gamma'$. A tensor mode in general does not perturb the ansatz, so neither should the particular scalar modes we are interested in (since they should satisfy the same equation). This leads to huge simplifications in (3.3) and (3.4) for scalar modes by setting $\zeta = 0$. The equations reduce to

$$\delta R_{GI} + (R_B + 3r_1) (\Phi - \Psi) = 0. \quad (3.36)$$

Substituting Φ and Ψ in terms of the scalar mode γ and dividing by the factor $-(R_B + 3r_1)/2$, we get

$$\ddot{\gamma} + \left(3H + \frac{\dot{R}_B}{R_B + 3r_1} \right) \dot{\gamma} = 0, \quad (3.37)$$

which is exactly the same as the evolution equation for tensor modes in the limit of $k \rightarrow 0$. We can conclude that homogeneous (i.e. $k \rightarrow 0$) and traceless perturbations on the metric diagonal consist of a scalar and a tensor mode and their evolution is described by (3.37). Thus all work on homogeneous perturbations of the form (3.35) fits into our more general framework, which uses a decomposition into scalars, vectors and tensors. Integrating (3.37), we can estimate the behavior of $\sigma^2 \equiv \sum_i \dot{\eta}_i^2$:

$$\sigma^2 \propto \frac{1}{a^6 (R_B + 3r_1)^2}. \quad (3.38)$$

In a bouncing context, scalar curvature is decreasing in the contracting phase such that anisotropies grow at least as a^{-6} . Bounce solutions in non-local gravity thus also suffer from a fine tuning problem (but not an infinite one, since there is a finite minimal scale factor).

4 Quantum perturbations

In standard inflation, cosmological perturbations have a quantum mechanical origin. At the beginning of inflation, modes of observational interest have wavelengths small compared to the Hubble radius. The standard assumption is that such modes start out in the Bunch-Davies vacuum.⁴ During inflation, their wavelengths become larger than the Hubble radius. Computation of the subsequent evolution is facilitated by the fact that certain variables

⁴We do not consider possible complications for trans-Planckian modes; see, for instance, [31, 32], for some discussion on this.

remain constant while they stay outside the horizon. Typically, inflationary theories predict (nearly) scale invariant power spectra of scalar and tensor modes after inflation. In this section, we discuss how perturbations can be quantized in non-local higher derivative gravity, at least in the de Sitter regime of the Starobinsky solution (which is our solution of most phenomenological interest). Restricting to the de Sitter regime amounts to focusing on physical wavelengths that satisfy

$$\frac{a}{k} \ll \frac{1}{\sqrt{r_1}}. \quad (4.1)$$

Indeed, as shown in section 7 of [16], such modes behave as if they propagated in de Sitter space. We calculate the power spectra of scalars and tensors as well as the tensor-to-scalar ratio.

Since we consider linear perturbations, we construct the second variation of the action, first in a covariant way. The metric is perturbed as $g_{\mu\nu} = g_{B\mu\nu} + h_{\mu\nu}$. We decompose $S = S_0 + S_1$ where

$$S_0 = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} R - \Lambda \right], \quad S_1 = \int d^4x \sqrt{-g} \frac{\lambda}{2} R \mathcal{F}(\square) R. \quad (4.2)$$

There is a problem computing $\delta^2 S_1$ around an inflationary background with a graceful exit because $(\delta \square) R_B \neq 0$, such that the non-local function \mathcal{F} itself is varied and we have no simple way of expressing the second variation of the action in terms of scalar modes. We are able to avoid this problem by focusing on modes that exit the horizon during the de Sitter phase of the Starobinsky solution, during which R_B is approximately constant and $R_{B\mu\nu} = R_B g_{\mu\nu}/4$.

For $\delta^2 S_0$ we use the results of [33]. The second variation is

$$\begin{aligned} \delta^2 S_0 = \int d^4x \sqrt{-g} \frac{M_P^2}{2} & \left[\frac{1}{4} h_{\mu\nu} \square_B h^{\mu\nu} - \frac{1}{4} h \square_B h + \frac{1}{2} h \nabla_{B\mu} \nabla_{B\rho} h^{\mu\rho} \right. \\ & \left. + \frac{1}{2} \nabla_{B\mu} h^{\mu\rho} \nabla_{B\nu} h_\rho^\nu - \frac{1}{48} R_B (h^2 + 2h_\nu^\mu h_\mu^\nu) \right] \equiv \int d^4x \sqrt{-g} \frac{M_P^2}{2} \delta_0. \end{aligned} \quad (4.3)$$

The variation of the non-local higher derivative action S_1 can now be easily expressed in terms of δR_{GI} and δ_0 ,

$$\delta^2 S_1 = \int d^4x \sqrt{-g} \left(\lambda \mathcal{F}(\square_B) R_B \delta_0 + \frac{\lambda}{2} \delta R_{GI} \mathcal{F}(\square_B) \delta R_{GI} \right), \quad (4.4)$$

where we have neglected terms containing the variation of the d'Alembertian operator because they are negligible compared to the second term of (4.4).⁵ Since the de Sitter background satisfies the ansatz (2.6), the total action quadratic in metric perturbations reduces to

$$\delta^2 S = \int d^4x \sqrt{-g} \frac{\lambda}{2} (2\mathcal{F}_1(R_B + 3r_1) \delta_0 + \delta R_{GI} \mathcal{F}(\square_B) \delta R_{GI}), \quad (4.5)$$

⁵To see this, one uses equations (3.4) and (3.2) of [25], combined with the fact that $\delta \square R_B$ is negligible compared to $(\square_B - r_1) \delta R_{GI}$. The latter statement can be verified by expressing both quantities in terms of the Bardeen potentials Φ and Ψ , and using (2.15), (4.1), $\dot{H}/H^2 \ll 1$ and $\dot{R}_B/H^3 \ll 1$.

4.1 Quantizing scalar modes

We consider scalar perturbations and work in the Newtonian gauge,

$$ds^2 = -a^2 [1 + 2\Phi] d\tau^2 + a^2 [1 - 2\Psi] dx^i dx_i. \quad (4.6)$$

Although the line element is written in terms of the gauge invariant Bardeen potentials, we have to be aware that the gauge is really fixed, namely that $\Phi = \phi$ and $\Psi = \psi$. In the Newtonian gauge the variation δ_0 is equal to

$$\delta_0 = -\frac{1}{a^2} \left[4k^2 \Psi^2 + 4k^2 \Psi \Phi + 12\mathcal{H} \Phi \Psi' + 6\mathcal{H}^2 \Phi^2 \right] - 6\Psi \square \Psi. \quad (4.7)$$

The Einstein equations lead to a further reduction of the action. They can be categorized into two types: dynamical equations and constraint equations [34, 35]. The latter contain at most first order time derivatives of the perturbed variables. They relate the Bardeen potentials to each other and determine constraints on the initial hypersurface on which quantisation proceeds. As done in [16] for a local higher derivative model, we use here the 0i equation (3.20) as a constraint. After a lengthy computation, the second variation of the action then reduces to

$$\delta^2 S = \int d^4 x \sqrt{-g} \frac{\lambda}{2\mathcal{F}_1 R_B} u \mathcal{W}(\square_B) (\square_B - r_1) \frac{1}{\mathcal{F}(\square_B)} u, \quad (4.8)$$

where total derivatives have been dropped and where we have taken into account that in the inflationary phase of the Starobinsky solution r_1 is negligible compared to R_B (to see this, use (3.17)). Varying with respect to u reproduces the perturbed trace equation.

Notice that we have silently assumed that $\mathcal{F}(\square)$ is invertible, which is only the case if \mathcal{F} has no roots. Moreover, the non-local theory should be ghostfree even at the perturbed level. This means that also \mathcal{W} must not have any roots.⁶ Indeed, a Weierstrass decomposition of \mathcal{W}/\mathcal{F} into its roots shows that every root introduces a classical degree of freedom. As soon as more than one root is present, the theory suffers from ghosts. The only root that is always present is the one which does not perturb the ansatz, so the one which is also present in $R + R^2$ gravity. The action can then be written as

$$\delta^2 S = \frac{1}{2} \int d^4 x \sqrt{-g} u e^{\gamma(\square_B)} \square_B u, \quad (4.9)$$

where

$$e^{\gamma(\square_B)} = \frac{\lambda \mathcal{W}(\square_B)}{\mathcal{F}_1 R_B \mathcal{F}(\square_B)} \quad (4.10)$$

and $\gamma(\square_B)$ is an entire function. A simple redefinition

$$\tilde{v} = e^{\gamma(\square_B)/2} u \quad (4.11)$$

turns the action into a local one

$$\delta^2 S = \frac{1}{2} \int d^4 x \sqrt{-g} \tilde{v} (\square_B - r_1) \tilde{v}. \quad (4.12)$$

⁶We notice here that in principle a strong coupling regime may occur if \mathcal{W} becomes small and thus it is implicitly assumed that \mathcal{W} is sufficiently large for our approximations to be reliable.

This local action in a de Sitter background can be easily written as

$$\delta^2 S = \frac{1}{2} \int d^4 x \left[v'^2 + v \Delta v + (2\mathcal{H}^2 - a^2 r_1) v^2 \right], \quad (4.13)$$

where

$$v = a \tilde{v}. \quad (4.14)$$

We retrieve the action of a scalar particle with time-dependent mass in Minkowski space, which can be straightforwardly quantized.

We want to compute the power spectrum $|\delta_\Phi(\vec{k}, \tau)|^2$, which is determined by the equal-time two point correlation function of the Bardeen potential Φ by

$$\langle 0 | \hat{\Phi}(\vec{x}, \tau) \hat{\Phi}(\vec{x} + \vec{r}, \tau) | 0 \rangle = \int_0^{+\infty} \frac{dk}{k} \frac{\sin(kr)}{kr} |\delta_\Phi(\vec{k}, \tau)|^2. \quad (4.15)$$

Here, the de Sitter invariant vacuum $|0\rangle$ is obtained by selecting negative frequency modes via the initial conditions

$$\begin{aligned} v(\vec{k}, \tau_0) &= \frac{1}{k^{3/2}} (\mathcal{H}_0 + ik) e^{ik\tau_0}, \\ v'(\vec{k}, \tau_0) &= \frac{i}{k^{1/2}} \left(\mathcal{H}_0 + ik - \frac{i\mathcal{H}_0'}{k} \right) e^{ik\tau_0}, \end{aligned} \quad (4.16)$$

where \mathcal{H}_0 is the Hubble constant at an early time τ_0 .

First of all we need to relate the two-point correlator of v to that of the Bardeen potential. The Einstein equations in de Sitter space allow us to relate Φ , Ψ and u . Combining (3.20) with (3.19) leads to the constraints

$$\Psi = \frac{1}{2\mathcal{F}_1 R_B} u, \quad \Phi = -\Psi, \quad (4.17)$$

where the last equality has been obtained by also using the ij component. Using (4.11) and (4.14), we find that

$$\begin{aligned} \langle 0 | \hat{\Phi}(\vec{x}, \tau) \hat{\Phi}(\vec{x} + \vec{r}, \tau) | 0 \rangle &= \frac{1}{(2\mathcal{F}_1 R_B)^2} \langle 0 | e^{-\gamma(\square_B)/2} \left(\frac{1}{a} \hat{v}(\vec{x}, \tau) \right) \\ &\quad \times e^{-\gamma(\square_B)/2} \left(\frac{1}{a} \hat{v}(\vec{x} + \vec{r}, \tau) \right) | 0 \rangle \\ &= \frac{1}{4\pi^2} \frac{1}{(2\mathcal{F}_1 R_B)^2} \int_0^{+\infty} \frac{dk}{k} \frac{\sin(kr)}{kr} k^3 \left| e^{-\gamma(\square_B)/2} \tilde{v}(\vec{k}, \tau) \right|^2. \end{aligned}$$

Here \hat{v} is the quantum field representing the canonically quantized classical field v . The vacuum state is defined at some initial time τ_0 corresponding to the beginning of inflation such that all initial perturbations are sub-Hubble. In the inflationary context, like the Starobinsky background, the solution for v implies that $\tilde{v}(k, \tau)$ is an eigenstate of the d'Alembertian with eigenvalue r_1 . It is given by (3.13) for $\nu \approx 3/2$. For such a mode, the initial conditions hold

not only at time τ_0 but at any time in the inflationary regime (i.e. as long as $r_1 \ll R_B$) because it is an approximate solution to the equation of motion for v (which can be easily derived from the action (4.13) after writing everything in Fourier space).

The primordial power spectrum of the sub-Hubble modes becomes

$$\left| \delta_\Phi(\vec{k}, \tau) \right|^2 \approx \frac{k^2}{16\pi^2 a^2} \frac{1}{\lambda \mathcal{F}_1 R_B} \frac{1}{3}, \quad (4.18)$$

where we re-expressed $e^{-\gamma(r_1)}$ through \mathcal{F} and \mathcal{W} and used (3.11) to obtain⁷ $\mathcal{F}_1/\mathcal{W}(r_1) = 1/3$. Like in a local higher derivative theory, the primordial spectrum depends on the physical wavelength as $|\delta_\Phi|^2 \propto 1/\lambda_{ph}^2$ where $\lambda_{ph} \sim a/k$. Since (4.16) holds at any time for a nearly massless mode, the spectrum in the long wavelength limit can be found by just taking the limit $k \ll \mathcal{H}$ of (4.16). In that way we retrieve the following scale invariant power spectrum:

$$\left| \delta_\Phi(\vec{k}, \tau) \right|^2 \approx \frac{H^2}{16\pi^2} \frac{1}{\lambda \mathcal{F}_1 R_B} \frac{1}{3}. \quad (4.19)$$

As a consistency check we note that our results for the power spectra reproduce the well known answers of $R + R^2$ gravity in a de Sitter limit [16]. It is so since we have reduced our consideration to the regime of the pure de Sitter inflation when the effects of non-localities vanish.

Usually in $R + R^2$ gravity one does not calculate the spectrum of Φ but rather that of

$$\mathcal{R} \equiv \Psi + \frac{H}{\dot{R}_B} \delta R_{GI}. \quad (4.20)$$

During the de Sitter phase of the Starobinsky solution, we can make use of (4.17), fill in that $u = \mathcal{F}(\square) \delta R_{GI}$ and remember that δR_{GI} satisfies (3.12) with eigenvalue r_1 and $\dot{H} \ll H^2$ such that \mathcal{R} reduces to

$$\mathcal{R} \approx -\frac{H^2}{\dot{H}} \Phi. \quad (4.21)$$

The advantage of calculating the spectrum of \mathcal{R} is that during a slow-roll inflation this quantity is constant in the super-Hubble limit. During the Hubble radius crossing we thus find a scale invariant primordial spectrum

$$\left| \delta_{\mathcal{R}}(\vec{k}, \tau) \right|^2 \approx \frac{H_{k=Ha}^6}{16\pi^2 \dot{H}_{k=Ha}^2} \frac{1}{\lambda \mathcal{F}_1 R_B} \frac{1}{3}. \quad (4.22)$$

4.2 Quantizing tensor modes

Since tensor perturbations behave in a more local way than scalar perturbations, we expect it should be possible to quantize them around more general spacetimes than de Sitter. However, in order to be able to compare power spectra of tensors and scalars, we will restrict our attention to de Sitter.

Using the fact that $\delta R_{GI} = 0$ under tensor perturbations, the second variation (4.5) of the action under tensor perturbations reads

$$\delta^2 S = \int d^4 x \sqrt{-g} \lambda \mathcal{F}_1 \xi \left(\frac{1}{4} h_{\mu\nu} \square_B h^{\mu\nu} - \frac{1}{24} R_B h_\nu^\mu h_\mu^\nu \right), \quad (4.23)$$

⁷The fact that the second term in (3.11) is zero when \mathcal{W} is evaluated at r_1 easily follows from (2.7).

where $\xi = R_B + 3r_1$, which in the de Sitter phase of the Starobinsky solution reduces to $\xi \approx R_B$. Tensors only perturb the spatial 3×3 part of the metric such that we can write $h_{\mu\nu} = a^2 h_{ij} \delta_{i\mu} \delta_{j\nu}$ and $h^{\mu\nu} = h_{ij} \delta_i^\mu \delta_j^\nu / a^2$. Furthermore, in a flat FLRW universe, $\sqrt{-g} = a^4$, from which it follows that

$$\delta^2 S = \int d^4 x \frac{\lambda \mathcal{F}_1 \xi a^2}{4} \left(h_{ij}'^2 + h_{ij} \Delta h_{ij} \right). \quad (4.24)$$

This is indeed the second variation of the action of $R + R^2$ gravity (with a cosmological constant) under tensor perturbations in a de Sitter background. In [16] the power spectrum of tensor perturbations in $R + R^2$ gravity was already calculated during a de Sitter stage. The quantization procedure is completely analogous to the scalar case, but with the difference that the action can now be written in such a way that we retrieve a massless mode in Minkowski space which describes a gravitational wave (see [16]).

The primordial power spectrum of sub-Hubble tensor modes in non-local higher derivative gravity is thus

$$|\delta_h|^2 = \frac{1}{\pi^2 2 \lambda \mathcal{F}_1 \xi} \frac{k^2}{a^2}. \quad (4.25)$$

In the super-Hubble regime, a scale invariant power spectrum is found:

$$|\delta_h|^2 = \frac{H^2}{\pi^2 2 \lambda \mathcal{F}_1 \xi}. \quad (4.26)$$

So, during a de Sitter stage, non-local higher derivative gravity predicts the same power spectrum as in $R + R^2$ gravity. Essentially this is due to the fact that tensors do not perturb the ansatz and therefore non-localities do not play a role.

By dividing the tensor and scalar power spectra, taking into account $\xi \approx R_B$ and including a factor of 2 because of the graviton polarizations, we retrieve

$$r = \frac{2 |\delta_h|^2}{|\delta_{\mathcal{R}}|^2} \approx 48 \frac{\dot{H}_{k=Ha}^2}{H_{k=Ha}^4} \quad (4.27)$$

for the tensor-to-scalar ratio r during the de Sitter phase of the Starobinsky solution. In terms of the slow-roll parameter $\epsilon_1 = - \left(\dot{H}/H^2 \right)_{k=Ha}$ the ratio becomes

$$r = 48 \epsilon_1^2. \quad (4.28)$$

The slow-roll parameter is easily related to the number of e -foldings $N \equiv \int_{t_i}^{t_f} H dt \approx 1/(2\epsilon_1)$ during the inflationary phase [36]. We now retrieve the familiar result

$$r = \frac{12}{N^2}. \quad (4.29)$$

Non-local higher derivative gravity thus predicts the same tensor-to-scalar ratio as $R + R^2$ gravity [36].

5 Summary and outlook

The original motivation for studying a class of non-local string field theory inspired generalizations of gravity (2.1), arose from the possibility of constructing a simple bouncing solution for the scale factor of an FLRW universe without introducing ghosts. In this paper we have shown the existence of an inflationary solution. This is the Starobinsky solution, which was originally found in $R + R^2$ gravity. Moreover, by noticing the classical equivalence with an $R + R^2$ model of gravity we have shown that every solution of $R + R^2$ gravity that is also a solution of a particular ansatz (2.6), solves the equations of motion of the non-local model. (This point has recently also been made in [37].) We have considered vector and tensor perturbations in non-local higher derivative gravity. Thanks to the fact that the action is built from scalar quantities only, we have managed to show that the equations of motion are perturbed under the vector and tensor perturbations in close analogy to $f(R)$ gravity models. The crucial technical consequence is that vector and tensor perturbations obey local equations of motion leaving all the non-localities at the background level. For special solutions that satisfy the simplifying ansatz (2.6), vector and tensor modes completely reduce to ones from $f(R)$ theories. We have illustrated this by considering their behavior around both a bouncing background and the Starobinsky solution. Both solutions have a de Sitter phase, enabling us to compare the behavior of these modes with that in standard inflationary theories. As a consequence of the equivalence with perturbations in $f(R)$ gravity, vector modes die out with time in an exponentially expanding phase. Non-local gravity therefore predicts no cosmological vector modes. In a bouncing context, tensor modes start growing in the contracting phase, pass the bounce smoothly and get frozen in at late times of the expanding phase. In the vicinity of bounce both vector and tensor modes grow, but not without bound. In an inflationary context, tensor modes are constant to leading order. In accordance with other inflationary models, non-local higher derivative gravity thus predicts the existence of cosmological gravitational waves.

Primordial metric perturbations are typically quantum mechanical fluctuations generated during inflation. In this paper we have shown how to quantize both scalar and tensor perturbations around the de Sitter phase. In this approximation, we have found that the power spectra for scalar and tensor modes during inflation agree with those in $R + R^2$ gravity.

A natural direction for future research in this framework is to consider the more general model introduced in [22, 38], the action of which contains not only the Ricci scalar, but also the Ricci tensor. In this model, gravity can be asymptotically free for a certain class of parameters. The model allows the same cosine hyperbolic solution, found using a generalized ansatz involving also the Ricci tensor. We expect that in this model the equations for vector and tensor modes will not reduce to those in $f(R)$ gravity, and that one will have to face non-local equations for vector and tensor perturbations.

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